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Reaction-diffusion processes in zero transverse dimensions as toy models for high-energy QCD

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Abstract: We examine numerically different zero-dimensional reaction-diffusion processes as candidate toy models for high-energy QCD evolution. Of the models examined — Reggeon Field Theory, Directed Percolation and Reversible Processes — only the latter shows the behaviour commonly expected, namely an increase of the scattering amplitude with increasing rapidity. Further, we find that increasing recombination terms, quantum loops and the heuristic inclusion of a running of the couplings, generically slow down the evolution.

Keywords: Stochastic Processes, QCD.

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1. Introduction

The seminal work of Munier and Peschanski [1] established a solid link between high energy QCD evolution and the physics of statistical mechanics systems. In that and ensuing works [2, 3], the connection between the Balitsky-Kovchegov (BK) equation [4, 5] and the Fisher-Kolmogorov-Petrovsky-Piscounov (FKPP) equation [6, 7], used to describe the time evolution of certain quantities in some statistical mechanics systems, was exploited to give a general and elegant derivation of the energy dependence of the saturation scale and of geometric scaling [8, 9].

The BK equation is the mean field limit of the more general set of evolution equations — the B-JIMWLK[1] equations [4, 10–16] — describing the evolution of QCD scattering amplitudes at high energy, i.e. the dynamics of the Colour Glass Condensate (CGC). Most studies of high energy QCD evolution have been performed using the BK equation. The reasons behind such preference are clear. Not only is the BK equation considerably more tractable than the B-JIMWLK set, but also the differences between the full B-JIMWLK evolution scheme and its truncated BK version are, perhaps surprisingly, small. The detailed numerical evaluation of these differences was performed in [17] by recasting the full evolution as a random walk in some functional space, once again highlighting the possible deep connection between QCD evolution and statistical mechanics.

The B-JIMWLK evolution equations were derived under the explicit assumption of the scattering of a dilute projectile on a dense target and should, therefore, be applicable only

[1] Balitsky-Jalilian-Marian-Iancu-McLerran-Weigert-Leonidov-Kovner.

for that physical situation. Such a limitation renders the CGC framework — either in its simplified BK version or for the more complete B-JIMWLK set — insufficient to address the experimentally important collisions of two nuclei. From a more theoretical perspective, this incompleteness of B-JIMWLK has been shown to eventually lead to the violation of unitary constraints [18, 19].

The inclusion of the missing ingredients — variously referred to as pomeron loops, fluctuations or wave function saturation effects — in an extended B-JIMWLK scheme has been attempted by several groups. A rather stringent constraint on the form of the complete evolution is imposed by what has been called the Dense-dilute Duality (DdD) [20,21]. This result, which establishes the equivalence of the dilute projectile, dense target case addressed by B-JIMWLK and its mirror image of a dense projectile and a dilute target, implies that the complete evolution kernel should necessarily be self-dual under the DdD.

Of particular relevance to the subject matter of this paper are those attempts [22–35] to include the missing effects which resort to possible generalizations of the correspondence of high energy QCD and statistical mechanics beyond the proven instance of [1]. It has been argued [22] that the incompleteness of B-JIMWLK can be related to the breakdown of the FKPP description in systems involving a small number of objects. Thus, a powerful tool for the generalization of B-JIMWLK could be provided by insights from statistical mechanics systems where the discreteness effects that violate the FKPP description can be accounted for via the inclusion of stochastic effects.

In the context of statistical mechanics systems, the failing FKPP description is remedied by the inclusion of a stochastic (noise) term accounting for the effects of number fluctuations which become increasingly important with decreasing number of intervening objects. Different choices for the stochastic term to be added to the FKPP equation will, in general, lead to different dynamical behaviours. The choice of the noise term is thus a crucial step in obtaining a suitable description of high energy QCD evolution in the framework of stochastic processes.

In general, there are two alternative strategies that can be followed to determine the form of the noise term. One can attempt to extend the known B-JIMWLK evolution equations by including the missing effects and then interpret the resulting equations in the stochastic language. This approach was followed in [25,28] where the obtained noise terms were, unfortunately, far too complex to allow any computation — analytical or numerical — to be carried out. Instead, one can choose to make an educated guess as to what the noise should be. Ultimately, for any such choice, the behaviour of the resulting Langevin equation will need to be compared with our general expectations for high energy evolution. In this paper we follow this latter strategy.

Our analysis is performed in the general framework of reaction-diffusion processes, in particular in the Hamiltonian language of [36], which is equivalent to the formulation in terms of Langevin equations. A Langevin equation in which the stochastic term is such that the functional form of the two-point noise correlation function replicates that of the non-diffusive part of the FKPP equation is usually referred to as the stochastic FKPP (sFKPP) equation and describes the dynamics of a reaction-diffusion process belonging [38] to the universality class of the reversible processes. If the noise is chosen to have a two-point

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correlator simply proportional to the quantity obeying the FKPP equation, the resulting Langevin equation belongs to a different universality class, that of directed percolation (for a comprehensive review see [37]). Although other choices for the form of the stochastic term are certainly possible, we concentrate on these two since they have been previously considered as possible descriptions of high energy QCD in the framework of stochastic processes.

We explore numerically the universality classes of reaction-diffusion processes with zero transverse dimensions. The Hamiltonian approach of [36] allows us to address both reversible processes and directed percolation universality classes within a single formulation. Further, the same formulation is well suited to enable a discussion of Reggeon Field Theory — both in its original form and with the extra quartic vertex recently considered in the context of high energy QCD evolution [34,39] — which, although related to directed percolation [40], does not correspond to a well defined reaction-diffusion process. Finally, it also allows for a rather straightforward heuristic introduction of 'running coupling effects'. Zero-dimensional toy models are thus used as a test bed for the far more involved realistic case with two transverse dimensions. By comparing the dynamical behaviour of the reaction-diffusion models with general expectations for high energy QCD evolution — geometrical scaling in BK, increase of the scattering amplitude with increasing rapidity, slowdown of evolution once couplings are allowed to run, diffusive scaling once stochastic effects are included — we narrow down the number of possible candidate models. The central result of this paper is that, at least in relation to toy models with zero transverse dimensions, the only universality class of reaction-diffusion processes compatible with high energy QCD evolution is that of reversible processes.

A word of caution is in order: high-energy QCD dynamics occurs in two transverse dimensions, so some features of our study may not correspond to the physical situation. Indeed, for the case of the reversible processes that we will study below, with critical dimension 2, the 0-dimensional case could not be a realistic approximation to high-energy QCD. Nevertheless, the simplicity of 0-dimensional models makes it possible to examine a wide variety of universality classes both exactly and approximately, and implement several variations in the models. So we find it worth to see which consequences can be extracted from 0-dimensional models with different reaction pieces, both to see their limitations as toy models and to compare the impact of some heuristic modifications with recent works in the literature. We hope that the conclusions extracted here in the restricted frame of 0-dimensional models, may be of help to constraint the eventual relation between QCD and reaction-diffusion processes.

In section 2 we introduce the formalism of [36], the toy models to be studied, an heuristic implementation of running of the couplings, and the classical solution of the problem. Our results are presented in section 3 and a concluding discussion in section 4.

2. Formalism

In this section we start by presenting the basic formalism that will be used to discuss the different classes of reaction-diffusion models. Then we motivate a choice of parameters

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in the models, discuss an heuristic introduction of the running of the couplings, and the classical solutions to the problem.

2.1 Basic setup

We use the formalism developed in [36]. In this reference, the diffusive term in a stochastic equation plays the role of a kinetic term in a Hamiltonian formulation, while the details of the process are encoded in a reaction Hamiltonian which corresponds to a potential. Following the notation of [36], the reaction Hamiltonian for an elementary reaction $kA \xrightarrow{\lambda} mA$, where λ is the strength of the reaction, reads

$$H_R(\bar{p},\bar{q}) = \frac{\lambda}{k!}(\bar{p}^m - \bar{p}^k)\bar{q}^k \,, \tag{2.1}$$

with $[\bar{q},\bar{p}] = 1$.

Instead of discussing arbitrary cases, we focus in this paper in Hamiltonians with two powers of $\bar{q}$ at most. While higher vertices might be required by an eventual 'true' theory of high energy QCD, we restrict to this form which contains the vertices most often discussed in the literature. To pass to a more usual notation, we make the changes $\bar{p} \to -q$, $\bar{q} \to p$, $H_R \to -H$, arriving at

$$H(p,q) = \alpha_1 pq - \alpha_2 qp^2 - \alpha_3 q^2 p + \alpha_4 q^2 p^2 \,. \tag{2.2}$$

This expression corresponds to the Hamiltonian which rules, for reactions involving vertices up to fourth order, the evolution in rapidity y (the logarithm of the energy) of an auxiliary function $F(y,q)$ [39],

$$\frac{\partial F(y,q)}{\partial y} = -H(p,q)F(y,q) \,. \tag{2.3}$$

The evolution starts from some functional form for the coupling to the projectile at $y = 0$, e.g. an eikonal coupling

$$F(y=0,q) = 1 - \exp\left(-g_i q\right), \tag{2.4}$$

where g_i is the coupling parameter with the projectile. The relation with the transition amplitude $A_{fi}(y)$ is given through

$$iA_{fi}(y) = F(y, q = g_f) \,, \tag{2.5}$$

where g_f is the coupling parameter with the target.

At this point a relation between the variable q and the dipole size can be argued. The dipole scattering amplitude on a hadronic target is known to increase monotonically with increasing dipole size. We assume in this study that the models that we are examining describe in a simplified manner the dynamics of fixed transverse size dipoles. In this way, the coupling to the target in (2.5), and thus the value of q, acquires a one-to-one correspondence with the dipole size. It is within this correspondence that we will introduce q as the scale for a running of the couplings in Subsection 2.3 and confront the results of the evolution of function $F(y,q)$ with the general expectation in QCD of increasing scattering amplitude with increasing dipole size.

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In (2.2) the variable p plays the role of a conjugate variable to q, $p = -\partial/\partial q \equiv -\partial_q$. It is, in the formalism of [36], the auxiliary degree of freedom required to study the non-equilibrium problem (for other approaches see [41–44]). Further, $q(p)$ can be interpreted as the creation (annihilation) operator of the objects exchanged in the scattering. Within this interpretation, the property of the Hamiltonian (2.2),

$$H(p, \partial_p) = H_{y \to -y}\left(\frac{\alpha_3}{\alpha_2} q, -\frac{\alpha_2}{\alpha_3}\partial_q\right) , \tag{2.6}$$

amounts to a symmetry of the scattering process from the projectile and target points of view. This symmetry is strongly suggestive of the Dense-dilute Duality which has been postulated [20] to be an essential property of high energy QCD evolution Hamiltonians.

The phase portrait of the different reaction-diffusion processes, defined by the zero energy trajectories, $H_R(\bar{p}, \bar{q}) = 0$, plays an important role in their classification into universality classes. More specifically, in [36] a classification of reaction-diffusion processes is proposed on the basis of the shape of the phase portrait. In this way, those processes whose phase portraits can be deformed into each other, would belong to the same universality class. Following this idea, in this work we will focus in the following cases:

1. Reggeon Field Theory (RFT): $\alpha_4 = 0$, $\alpha_2 = \alpha_3$. It contains no quartic vertex and the splitting and recombination strengths are equal. It has no correspondence to a reaction-diffusion process, and it displays the known phenomenon that the amplitude vanishes with increasing rapidity which is usually interpreted as tunneling [45–47]. The effect of the quartic vertex has been examined recently [34] due to its possible connection to high energy QCD evolution. The zero energy lines determine a triangular phase portrait given by (see figure 1):

$$\bar{p} = 0, \quad \bar{q} = 0, \quad \bar{q} = \frac{\alpha_1}{\alpha_2} + \bar{p} \, . \tag{2.7}$$

2. Directed Percolation (DP): it is the case for the stochastic process with allowed reactions $1 \xrightarrow{\lambda} 0$, $1 \xrightarrow{\mu} 2$ and $2 \xrightarrow{2\sigma} 1$, leading to $\alpha_1 = \mu - \lambda$, $\alpha_3 = \mu$ and $\alpha_2 = \alpha_4 = \sigma$. It contains a quartic vertex with the same strength as the recombination vertex, and its phase portrait is determined by (see figure 2):

$$\bar{p} = 0, \quad \bar{q} = 0, \quad \bar{q} = \frac{\alpha_1 + \alpha_3 \bar{p}}{\alpha_2 (1 + \bar{p})} \, . \tag{2.8}$$

   Renormalization group arguments lead to the conclusion that the quartic vertex is irrelevant at dimensions greater than 4 (see e.g. [48]). On the other hand, the third trajectory in (2.8) goes into a straight line when expanded around $\bar{p} = 0$

$$\bar{q} \simeq \frac{\mu - \lambda}{\sigma} + \left.\frac{\mu\sigma(1+\bar{p}) - [(\mu-\lambda) + \mu\bar{p}]\sigma}{\sigma^2 (1+\bar{p})^2}\right|_{\bar{p}=0} \bar{p} = \frac{\mu - \lambda}{\sigma} + \frac{\lambda}{\sigma}\bar{p} \, . \tag{2.9}$$

   In both cases one returns to RFT, a fact that we will use below to fix the parameters for our numerical study.

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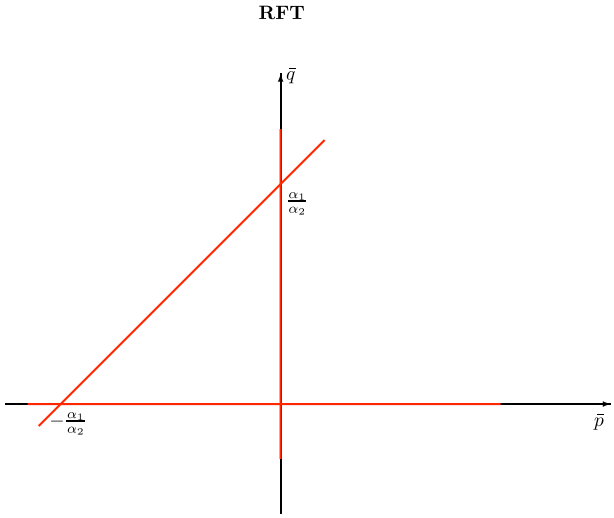

Figure 1: Phase portrait of RFT.

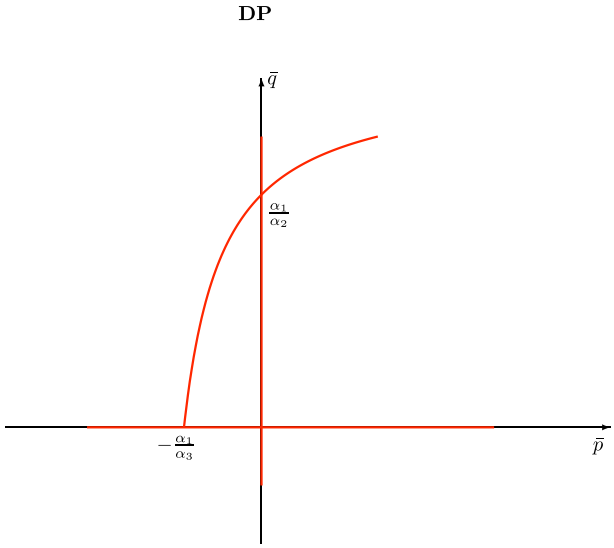

Figure 2: Phase portrait of DP.

3. Reversible processes (RP): it is the case for the stochastic process with reactions $1 \xrightarrow{\mu} 2$ and $2 \xrightarrow{2\sigma} 1$, leading to $\alpha_1 = \alpha_3 = \mu$ and $\alpha_2 = \alpha_4 = \sigma$. It contains a quartic vertex with the same strength as the recombination vertex and its phase portrait is determined by (see figure 3):

$$\bar{p} = 0, \quad \bar{q} = 0, \quad \bar{p} = -1, \quad \bar{q} = \frac{\alpha_1}{\alpha_2}\,. \tag{2.10}$$

This phase portrait cannot be deformed to a triangle and it does, therefore, belong to a different universality class than that of DP. The Langevin equation corresponding

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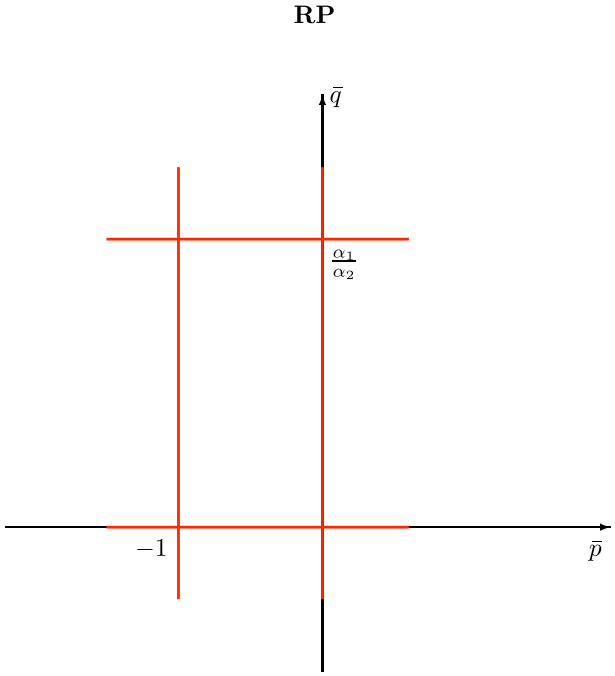

Figure 3: Phase portrait of RP.

to this reaction-difussion problem is the stochastic Fisher-Kolmogorov-Petrovsky-Piscounov (see recent analyses in [49, 50]), which has been proposed to play a role in high energy QCD evolution beyond JIMWLK [23, 25, 27, 35].

2.2 Choice of parameters

Instead of scanning a large region of the parameter space, we focus on the effect of the quartic vertex and of the difference between different reaction-diffusion processes. Thus, for RFT we choose values $\alpha_1 = 1$ (this parameter can always be absorbed in a redefinition of rapidity) and $\alpha_2 = \alpha_3 = 0.5$. We have checked that with this set of parameters the conclusions we extract by numerically solving the evolution equation (2.3) in a restricted rapidity window $0 < y < 5$ hold for higher rapidities and are not a 'subasymptotic' effect, a situation which may happen for smaller values of $\alpha_2 = \alpha_3$ [39]. Besides, we will generically study the introduction of a quartic vertex in RFT as done in [34] though this situation, as RFT itself in zero dimensions, does not correspond to any reaction-diffusion process.

For DP we adopt the following strategy: we approximate the phase portrait (2.8) by a triangle — see (2.9) — which brings DP into RFT with a given quartic vertex, and identify the parameters with those in the phase portrait of RFT (2.7). This leads to the values $\alpha_1 = 1$, $\alpha_2 = \alpha_4 = 0.5$ and $\alpha_3 = 1.5$.

Finally, for RP we take $\alpha_1 = \alpha_3 = 1$ and vary the remaining parameter in the region $0.1 < \alpha_2 = \alpha_4 < 0.9$.

At this point it is tempting to try to establish some relation between the four parameters in the reaction-diffusion models that we consider and QCD parameters. However,

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such an identification depends on the degrees of freedom, presently unknown, which could eventually make it possible to identify high-energy QCD with a reaction-diffusion process. Were the relevant QCD degrees of freedom dipoles, the proportionality, in RP, of couplings 1 and 3 to α_s and of couplings 2 and 4 to α_s^2, would be tempting. But in the dipole model the quartic vertex is parametrically suppressed by the number of colours, therefore preventing such a relation. Thus, we prefer to restrict ourselves to the RFT situation, where the four couplings are of the same order, and a comparison of the results of the evolution of the three models (RFT, DP and RP) is possible and given solely by their different properties as reaction-diffusion processes.

2.3 Running of the couplings

All couplings in these reaction-difussion processes are, at this level, fixed. Recently it has been proposed in the framework of a one-dimensional model, that effects of the running of the couplings may shift the contribution of effects of loops to higher rapidities [51]. To implement the running of the coupling in our approach we adopt an heuristic procedure. We interpret $1/q$ as some 'momentum' scale which determines a common logarithmic running for the four couplings in our Hamiltonian[2]:

$$\alpha_i(q) = \alpha_i \frac{\ln(Q/q)}{\ln(q_0/q)}, \quad i = 1, \ldots, 4\,, \tag{2.11}$$

for $q < q_0$ and frozen to the values α_i discussed in the previous Subsection for $q \geq q_0$. $Q = 10\,q_0$ plays the role of an inverse QCD scale, and we take two values of $q_0 = 0.5$ and 1. These choices are motivated by the need of making the effect of the running of the coupling noticeable in the restricted region of q we explore numerically.

2.4 Classical solutions

The Hamiltonian problem admits a classical solution:

$$\begin{aligned} \dot{p} &= (-\alpha_1 + \alpha_2 p)p + 2(\alpha_3 - \alpha_4 p)qp\,, \\ \dot{q} &= (\alpha_1 - 2\alpha_2 p)q + (-\alpha_3 + 2\alpha_4 p)q^2\,, \end{aligned} \tag{2.12}$$

with initial conditions

$$q(y=0) = g_i, \quad p(y=Y) = g_f\,, \tag{2.13}$$

where Y is the total rapidity spanned between the projectile located at $y = 0$ and the target located at $y = Y$. Thus, the classical amplitude, i.e. the amplitude at tree level, can be computed standardly [47, 53]:

$$iA_{fi}^{\text{clas}}(y) = 1 + \sum_k \Delta_k \exp\left[-S(Y, q_k, p_k)\right], \tag{2.14}$$

[2] As discussed below (2.5) we identify the amplitude with that for the scattering of a dipole of size $\propto q$, which is known to increase with increasing dipole size. Previous experience with running coupling effects in the BK equation [52] give us the hope that the concrete choice of scale will not alter the qualitative conclusions of this study.

with the index k running over all possible solutions of the classical equations of motion (2.12)[3], $\Delta_k = \pm 1$ and $S(Y, q_k, p_k)$ the action evaluated for the classical solutions.

3. Results

We begin this section with a description of the numerical method used to compute the transition amplitude. Following that, we study the rapidity evolution of the solutions for the fan case, RFT, DP and RP. Additionally, we examine the effects of an heuristic introduction of running of the couplings. Further, we phrase the effects of evolution in terms of a saturation scale. Finally, we compare the results of quantum and classical evolution.

3.1 Numerical method

To solve the differential equation (2.3) with evolution kernel (2.2) and initial conditions (2.4) which defines the quantum mechanical problem, we use a second order Runge-Kutta method. We have discretized the q-range in 500 points per unit which showed to be enough for a precision better than a few percent. The rapidity region studied is

$$0 \leq y \leq 5 \tag{3.1}$$

(though we study values as high as 40 when comparing with the classical solution). The step in rapidity is correlated with the step in q [39]; we have used $h = 6.25 \cdot 10^{-6}$. With this numerical method we obtain the values of $F_i(y, q)$ in a q interval for different rapidities. The results depends also on the coupling constants: we use $g_i = 1$ [39].

The solution to the classical equations of motion (2.12) with initial conditions (2.13) was obtained using the shooting method. After the classical trajectories were computed, the amplitude was calculated through (2.14) (the corresponding quantum expression is given by (2.5)).

3.2 Numerical results

We focus on the rapidity evolution of the solutions, confronting the results with the expectations coming from the correspondence of the variable q with a dipole size, as argued below (2.5)).

(a) fan case

First, as a check, we study the so-called fan case [54] in which there are no recombination terms, from the projectile point of view, in the Hamiltonian (2.2), i.e. $\alpha_2 = \alpha_4 = 0$. The differential equation admits an analytical solution

$$F_{\text{fan}}(y, q) = 1 - \exp\left[-\frac{g_i q e^{\alpha_1 y}}{1 + \frac{q\alpha_3}{\alpha_1}\left(e^{\alpha_1 y} - 1\right)}\right] . \tag{3.2}$$

[3]Three solutions, one symmetric ($\Delta_k = +1$) between projectile and target and two asymmetric ($\Delta_k = -1$), appear above some critical rapidity whose value depends on the chosen initial conditions, see a recent discussion in [53].

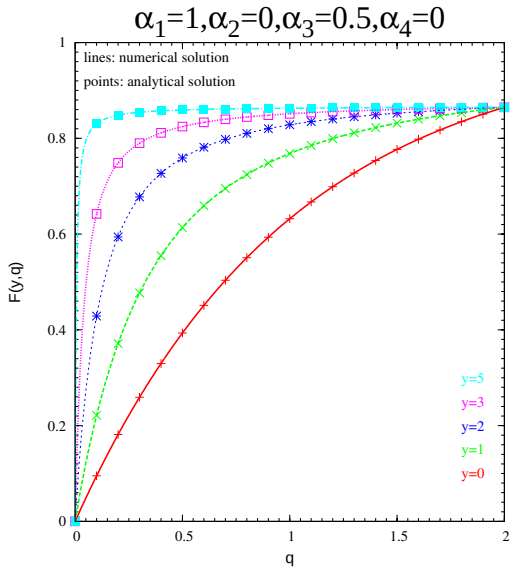

Figure 4: Comparison of the analytical (symbols) and numerical results (lines) for the fan case, with $g_i = 1$.

We use this analytical result to perform a check of our numerical solution. As we can see in figure 4 there is a very good agreement between the analytical and the numerical results. Note that the BK equation sums the corresponding fan diagrams for BFKL pomerons [55], leading a behaviour similar to what we found in this simple zero-dimensional case.

(b) RFT

Now we turn to RFT — figure 5 — which contains no quartic vertex. In the limit of high rapidity we find an exponential decay, a behaviour already found in [34, 39] and interpreted as a tunneling phenomenon [45–47].

(c) DP

The third case we study is DP, with the parameters fixed as discussed in Subsection 2.2. What we find, figure 6, is that the evolution goes in the opposite direction to what is expected in high energy evolution, i.e. that the function moves to smaller values of q with increasing rapidity [4]. This poses serious doubts on DP as a candidate for a description of high energy QCD evolution.

(d) RP

Finally we turn to RP. As we see in figure 7 (left), the evolution leads the front towards smaller values of q as we increase the value of rapidity. So it behaves in the expected manner for high energy QCD evolution. The parameters $\alpha_{2,4}$, characterizing

[4] Our choice of parameters makes this behaviour visible in the y-range we study. Smaller values of $\alpha_{2,3,4}$ make it noticeable only for larger values of y.

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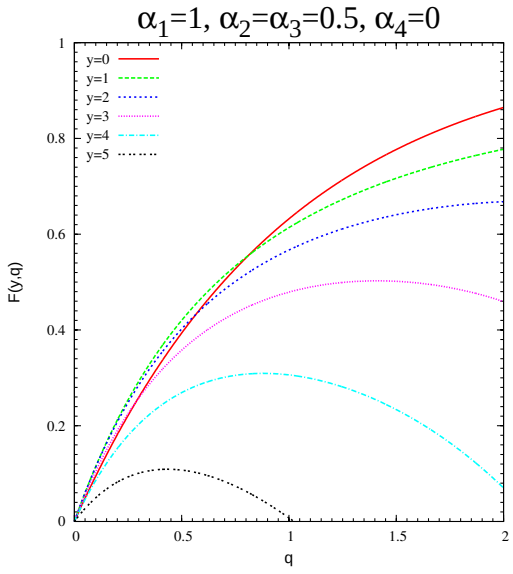

Figure 5: Rapidity evolution of the solutions in the case of RFT.

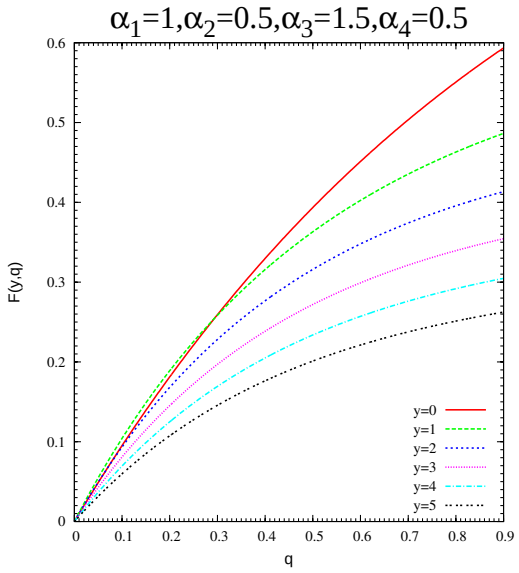

Figure 6: Rapidity evolution of the solutions in the case of DP.

respectively the vertices $2 \to 1$ and $2 \to 2$, are free. We find that increasing $\alpha_2 = \alpha_4$ makes the evolution softer — figure 7 (right) –, again as expected for high energy evolution.

(e) running coupling

Now we turn to our heuristic implementation of the running of the couplings. In figure 8 we show the results for the fan case and for RP — the two cases where the

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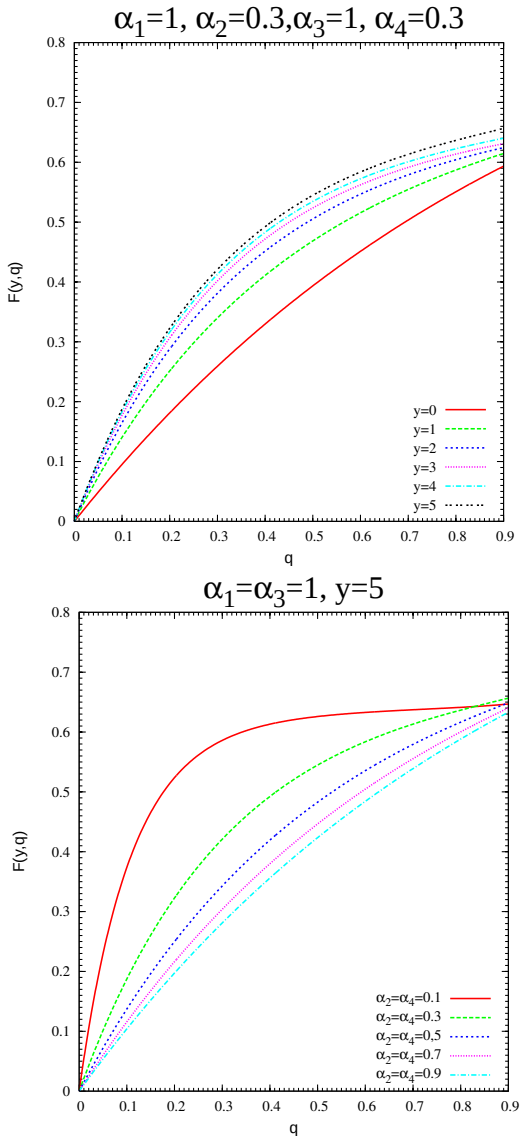

Figure 7: Rapidity evolution of the solutions in the case of RP for $\alpha_2 = \alpha_4 = 0.3$ (left). Solutions of RP for different values of $\alpha_2 = \alpha_4$, at $y = 5$ (right).

evolution goes in the expected direction — for the different freezing points for the coupling discussed in subsection 2.3. The evolution is clearly slowed down with the effect of the frozen procedure being more noticeable at the beginning of the evolution.

(f) saturation scale

To ensure that the effects of evolution in each case are more easily visible, we parallel the studies of the BK equation and introduce a 'saturation scale' $1/q_s(y)$, defined through the equation

$$F(y, q_s(y)) = \kappa\,, \tag{3.3}$$

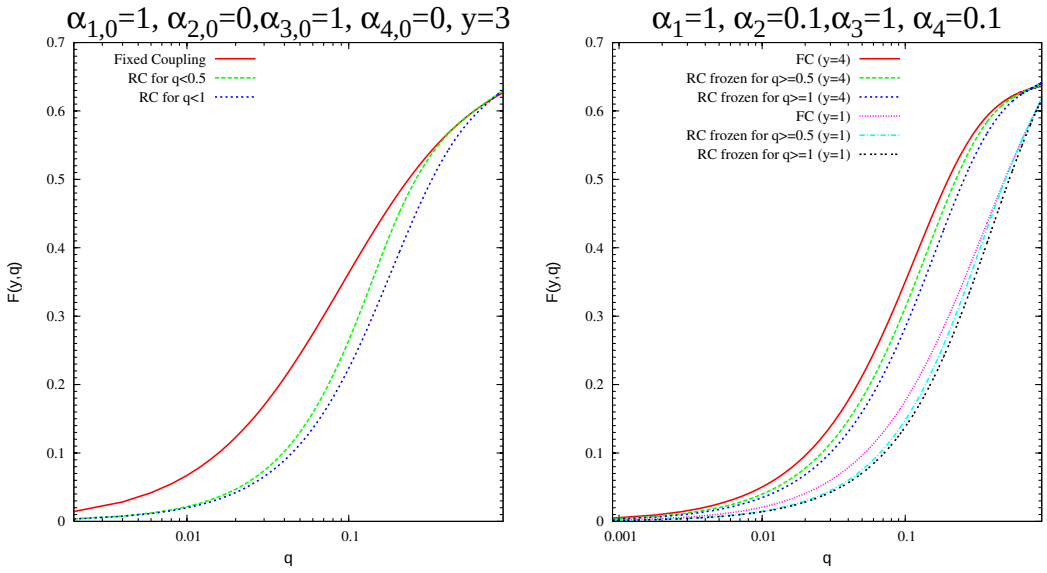

Figure 8: Evolution of $F_i(y,q)$ for given values of rapidity, comparing the behaviour with fixed (FC) and running (RC) couplings. Fan case (left). RP (right).

with κ some fixed value of order 1/2. It is known [52] that the specific choice of the value of κ leads to small changes in the leading y-behaviour of the saturation scale which, for the purposes of this study, are entirely negligible. In figure 9 we show the 'saturation scale' obtained for the fan case, for DP and for RP. In the fan case, the running of the couplings slows down the increase of $1/q_s(y)$ (as in the BK equation). In DP $1/q_s(y)$ diminishes with increasing rapidity. Finally, in RP both the increase of the strengths of the vertices $2 \to 1$ and $2 \to 2$ and the running of the couplings slow down the increase of $1/q_s(y)$ — a behaviour which may be linked with the findings in [51] of a competition between the effects of recombination vertices, called 'Pomeron loops', and of running coupling.

(g) classical solution

Finally, we want to compare the full quantum and the symmetrical classical solutions, see Subsection 2.4. This provides an evaluation of the effect of loops in the strict Quantum Field Theory sense. In figure 10 we show both the classical and quantum imaginary parts of the amplitude for RFT with a quartic vertex, for different values of $g_i = g_f$. The classical solution lies always above the quantum one, with the difference between both increasing with increasing rapidity. Similar conclusions are extracted in other cases.

4. Conclusions

In this paper we have examined numerically several reaction-diffusion processes in zero transverse dimensions as possible candidates for a toy model of high-energy QCD evolution.

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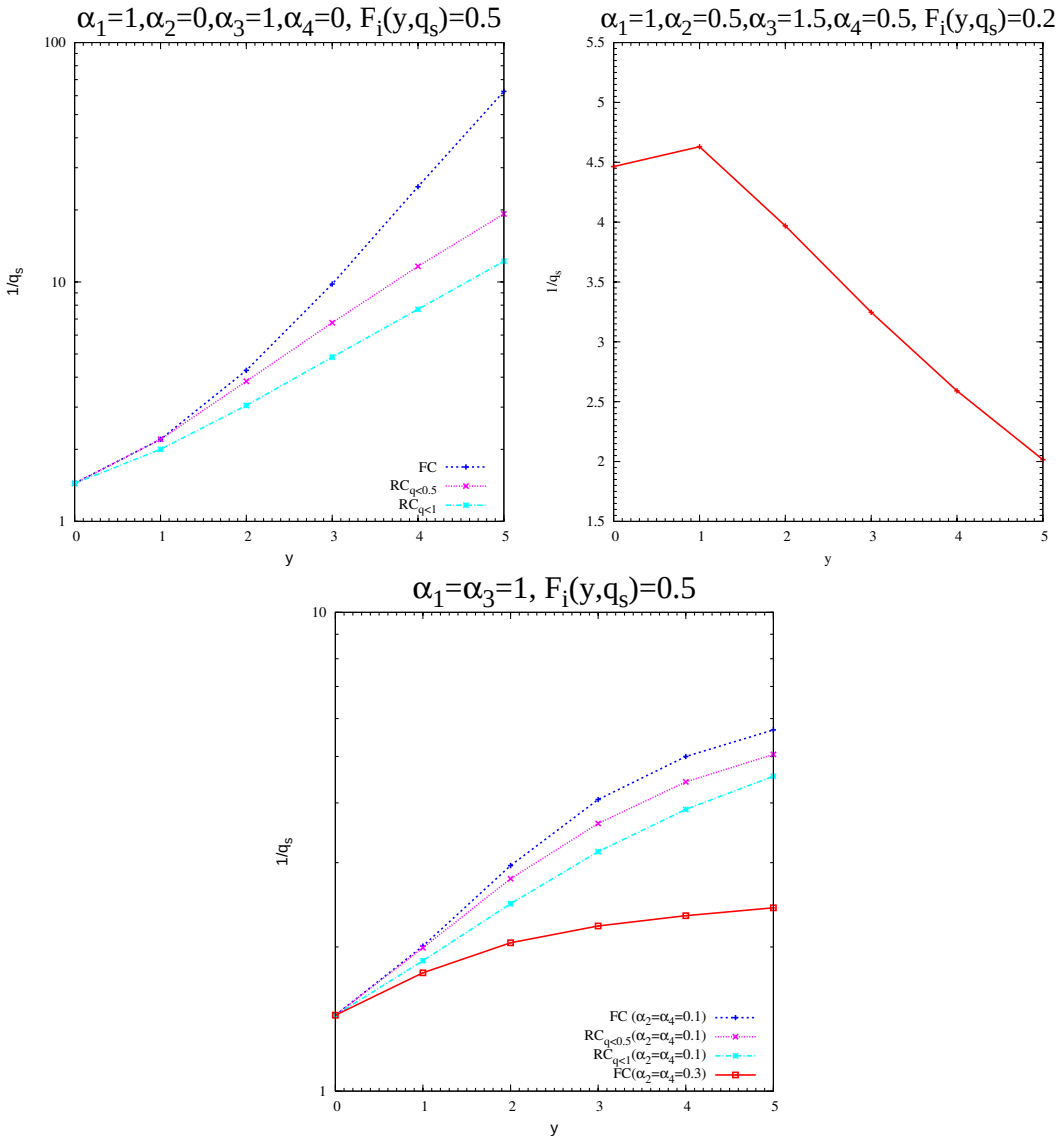

Figure 9: $1/q_s(y)$ in the fan case for fixed and running couplings (top left). $1/q_s(y)$ in DP for fixed coupling (top right). $1/q_s(y)$ in RP with fixed coupling for two sets of $\alpha_2 = \alpha_4 = 0.1$ and 0.3, and for running coupling for $\alpha_2 = \alpha_4 = 0.1$ (bottom).

We have restricted ourselves to those models in which the Hamiltonian formulation [36] contain vertices with up to two derivatives: diffusion, splitting, recombination and $2 \to 2$, which allows a discussion of all common universality classes. We have also considered usual variants of such models which are not strictly reaction-diffusion processes. Their feasibility as toy models for high-energy QCD evolution has been evaluated in terms of the direction of the evolution with increasing evolution parameter, that is increasing rapidity.

In the case of Reggeon Field Theory, which contains no quartic vertex and thus has no reaction-diffusion interpretation, we find the known behaviour of a vanishing amplitude

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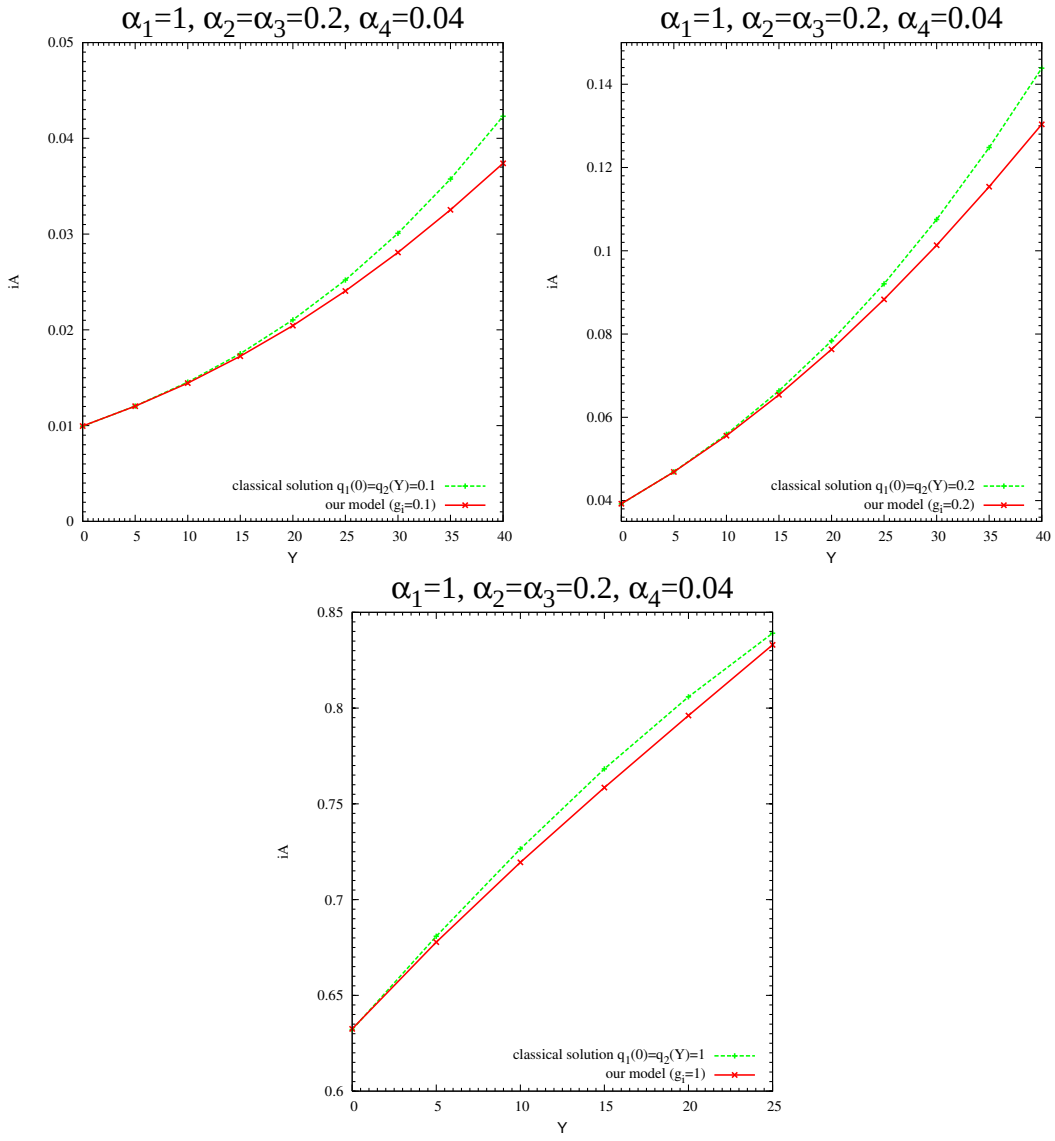

Figure 10: The classical symmetrical (dashed lines) and the quantum (solid line) imaginary part of the amplitude versus y. $g_i = g_f = 0.1$ (top left). $g_i = g_f = 0.2$ (top right). $g_i = g_f = 1$ (bottom).

with increasing rapidity. Directed Percolation leads to a behaviour of the solutions decreasing with increasing rapidity. On the other hand, the limiting case of considering only fan diagrams (again with no reaction-diffusion counterpart) and Reversible Processes show solutions which increase with increasing rapidity. In the case of RP which, as DP, contains a quartic vertex, the evolution is slowed down by increasing the recombination and $2 \to 2$ vertices. We have shown all these behaviours both at the level of the amplitudes and of an analogue of a saturation momentum.

We have also introduced, in an heuristic manner, a running of the couplings, which tends to slow down the evolution in all cases, working in the same direction as the increase

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of the recombination and $2 \to 2$ vertices as found in [51]. Finally, we have numerically computed the classical solutions (those which sum tree diagrams), and found that they generically lie above the quantum ones, with their difference growing with increasing rapidity, so quantum effects tend to slow down the evolution.

Summarizing, we have found that the only zero-dimensional reaction-diffusion process which shows a behaviour of the amplitude in agreement with the expectations from high energy QCD with rapidity is RP. This process is linked with sFKPP equation often used to discuss [23,25,27,35] high-energy QCD evolution. Increasing recombinations terms, and the inclusion of quantum loops and of a running of the coupling generically slow down the evolution.

The analysis of reaction-diffusion processes in zero transverse dimensions presented in this paper serves as a basis for future analogous studies involving a number of dimensions larger than zero, in particular for the two transverse dimension case relevant for QCD. In zero dimensions reaction-diffusion processes are purely reactive (i.e., the diffusion term is absent). The presence of a diffusion term — be it the full BFKL kernel as relevant for QCD or a 'toy' one dimensional kernel — will lead to changes of the evolution trends established in this work. For particular choices of the reaction part of the process, the diffusive term will lead to the appearance of a limiting maximal noise (in our language, a limiting value for the couplings) at which the evolution will cease. This situation has been recently examined in [56]. Future work will examine the behaviour of reaction-diffusion processes with a realistic BFKL diffusive kernel.

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References

[1] S. Munier and R.B. Peschanski, *Geometric scaling as traveling waves*, *Phys. Rev. Lett.* **91** (2003) 232001 [hep-ph/0309177].

[2] S. Munier and R.B. Peschanski, *Universality and tree structure of high energy QCD*, *Phys. Rev.* **D 70** (2004) 077503 [hep-ph/0401215].

JHEP05(2008)103

[3] S. Munier and R.B. Peschanski, *Traveling wave fronts and the transition to saturation*, *Phys. Rev.* **D 69** (2004) 034008 [hep-ph/0310357].

[4] I. Balitsky, *Operator expansion for high-energy scattering*, *Nucl. Phys.* **B 463** (1996) 99 [hep-ph/9509348].

[5] Y.V. Kovchegov, *Small-x F2 structure function of a nucleus including multiple Pomeron exchanges*, *Phys. Rev.* **D 60** (1999) 034008 [hep-ph/9901281].

[6] R.A. Fisher, *The advance of advantageous genes*, *Ann. Eugenics* **7** (1937) 355.

[7] A. Kolmogorov, I. Petrovsky and N. Piscounov, *Study of the diffusion equation, in conjunction with the increase in the quantity of matter and its application to a problem in biology*, *Moscow Gos. Univ. Bull.* **A1** (1937) 1.

[8] E. Iancu, K. Itakura and L. McLerran, *Geometric scaling above the saturation scale*, *Nucl. Phys.* **A 708** (2002) 327 [hep-ph/0203137].

[9] A.H. Mueller and D.N. Triantafyllopoulos, *The energy dependence of the saturation momentum*, *Nucl. Phys.* **B 640** (2002) 331 [hep-ph/0205167].

[10] L.D. McLerran and R. Venugopalan, *Computing quark and gluon distribution functions for very large nuclei*, *Phys. Rev.* **D 49** (1994) 2233 [hep-ph/9309289].

[11] L.D. McLerran and R. Venugopalan, *Gluon distribution functions for very large nuclei at small transverse momentum*, *Phys. Rev.* **D 49** (1994) 3352 [hep-ph/9311205].

[12] L.D. McLerran and R. Venugopalan, *Green's functions in the color field of a large nucleus*, *Phys. Rev.* **D 50** (1994) 2225 [hep-ph/9402335].

[13] J. Jalilian-Marian, A. Kovner and H. Weigert, *The Wilson renormalization group for low x physics: gluon evolution at finite parton density*, *Phys. Rev.* **D 59** (1999) 014015 [hep-ph/9709432].

[14] A. Kovner, J.G. Milhano and H. Weigert, *Relating different approaches to nonlinear QCD evolution at finite gluon density*, *Phys. Rev.* **D 62** (2000) 114005 [hep-ph/0004014].

[15] E. Iancu, A. Leonidov and L.D. McLerran, *Nonlinear gluon evolution in the color glass condensate. I*, *Nucl. Phys.* **A 692** (2001) 583 [hep-ph/0011241].

[16] E. Ferreiro, E. Iancu, A. Leonidov and L. McLerran, *Nonlinear gluon evolution in the color glass condensate. II*, *Nucl. Phys.* **A 703** (2002) 489 [hep-ph/0109115].

[17] K. Rummukainen and H. Weigert, *Universal features of JIMWLK and BK evolution at small x*, *Nucl. Phys.* **A 739** (2004) 183 [hep-ph/0309306].

[18] E. Iancu and A.H. Mueller, *From color glass to color dipoles in high-energy onium onium scattering*, *Nucl. Phys.* **A 730** (2004) 460 [hep-ph/0308315].

[19] A.H. Mueller and A.I. Shoshi, *Small-x physics beyond the Kovchegov equation*, *Nucl. Phys.* **B 692** (2004) 175 [hep-ph/0402193].

[20] A. Kovner and M. Lublinsky, *From target to projectile and back again: selfduality of high energy evolution*, *Phys. Rev. Lett.* **94** (2005) 181603 [hep-ph/0502119].

[21] A. Kovner and M. Lublinsky, *Dense-dilute duality at work: dipoles of the target*, *Phys. Rev.* **D 72** (2005) 074023 [hep-ph/0503155].

JHEP05(2008)103

[22] E. Iancu, A.H. Mueller and S. Munier, *Universal behavior of QCD amplitudes at high energy from general tools of statistical physics*, *Phys. Lett.* **B 606** (2005) 342 [hep-ph/0410018].

[23] E. Iancu and D.N. Triantafyllopoulos, *A Langevin equation for high energy evolution with Pomeron loops*, *Nucl. Phys.* **A 756** (2005) 419 [hep-ph/0411405].

[24] R.A. Janik, *QCD saturation in the dipole sector with correlations*, *Phys. Lett.* **B 604** (2004) 192.

[25] A.H. Mueller, A.I. Shoshi and S.M.H. Wong, *Extension of the JIMWLK equation in the low gluon density region*, *Nucl. Phys.* **B 715** (2005) 440 [hep-ph/0501088].

[26] E. Levin and M. Lublinsky, *Towards a symmetric approach to high energy evolution: generating functional with Pomeron loops*, *Nucl. Phys.* **A 763** (2005) 172 [hep-ph/0501173].

[27] E. Iancu and D.N. Triantafyllopoulos, *Non-linear QCD evolution with improved triple-Pomeron vertices*, *Phys. Lett.* **B 610** (2005) 253 [hep-ph/0501193].

[28] E. Iancu, G. Soyez and D.N. Triantafyllopoulos, *On the probabilistic interpretation of the evolution equations with Pomeron loops in QCD*, *Nucl. Phys.* **A 768** (2006) 194 [hep-ph/0510094].

[29] C. Marquet, R.B. Peschanski and G. Soyez, *Consequences of strong fluctuations on high-energy QCD evolution*, *Phys. Rev.* **D 73** (2006) 114005 [hep-ph/0512186].

[30] N. Armesto and J.G. Milhano, *On correlations and discreteness in non-linear QCD evolution*, *Phys. Rev.* **D 73** (2006) 114003 [hep-ph/0601132].

[31] J.P. Blaizot, E. Iancu and D.N. Triantafyllopoulos, *A zero-dimensional model for high-energy scattering in QCD*, *Nucl. Phys.* **A 784** (2007) 227 [hep-ph/0606253].

[32] S. Munier, *Dense-dilute factorization for a class of stochastic processes and for high energy QCD*, *Phys. Rev.* **D 75** (2007) 034009 [hep-ph/0608036].

[33] E. Iancu, J.T. de Santana Amaral, G. Soyez and D.N. Triantafyllopoulos, *One-dimensional model for QCD at high energy*, *Nucl. Phys.* **A 786** (2007) 131 [hep-ph/0611105].

[34] S. Bondarenko, L. Motyka, A.H. Mueller, A.I. Shoshi and B.W. Xiao, *On the equivalence of Reggeon field theory in zero transverse dimensions and reaction-diffusion processes*, *Eur. Phys. J.* **C 50** (2007) 593 [hep-ph/0609213].

[35] S. Bondarenko, *Langevin equation in effective theory of interacting QCD Pomerons in the limit of large-N_c*, *Nucl. Phys.* **A 792** (2007) 264 [hep-ph/0703045].

[36] V. Elgart and A. Kamenev, *Classification of phase transitions in reaction-diffusion models*, *Phys. Rev.* **E 74** (2006) 041101 [cond-mat/0605041].

[37] H.K. Janssen and U. Täuber, *The field theory approach to percolation processes*, *Ann. Phys. (NY)* **315** (2005) 147 [cond-mat/0409670].

[38] V. Doering, C. Mueller and P. Smereka, *Interacting particles, the stochastic Fisher-Kolmogorov-Petrovsky-Piscounov equation, and duality*, *Physica* **A 325** (2003) 243.

[39] M.A. Braun and G.P. Vacca, *On the 'toy model' in the Reggeon field theory*, *Eur. Phys. J.* **C 50** (2007) 857 [hep-ph/0612162].

[40] J.L. Cardy and R.L. Sugar, *Directed percolation and Reggeon field theory*, *J. Phys.* **A 13** (1980) L423.

JHEP05(2008)103

[41] M. Doi, *Stochastic theory of diffusion-controlled reaction*, *J. Phys.* **A 9** (1976) 1479.

[42] L. Peliti, *Path integral approach to birth-death processes on a lattice*, *J. Physique* **46** (1985) 1469.

[43] H.K. Janssen, *Lagrangean for classical field dynamics and renormalization group calculations of dynamical critical properties*, *Z. Phys.* **B23** (1976) 377.

[44] P.C. Martin, E.D. Siggia and H. . Rose, *Statistical dynamics of classical systems*, *Phys. Rev.* **A 8** (1978) 423.

[45] V. Alessandrini, D. Amati and R. Jengo, *One-dimensional quantum theory of the Pomeron*, *Nucl. Phys.* **B 108** (1976) 425.

[46] M. Ciafaloni, M. Le Bellac and G.C. Rossi, *Reggeon quantum mechanics: a critical discussion*, *Nucl. Phys.* **B 130** (1977) 388.

[47] M. Ciafaloni, *Instanton contributions in Reggeon quantum mechanics*, *Nucl. Phys.* **B 146** (1978) 427.

[48] U.C. Tauber, M. Howard and B.P. Vollmayr-Lee, *Applications of field-theoretic renormalization group methods to reaction-diffusion problems*, *J. Phys.* **A 38** (2005) R79 [cond-mat/0501678].

[49] R. Jack, P. Mayer and P. Sollich, *Mappings between reaction-diffusion and kinetically constrained systems: $A + A \leftrightarrow A$ and the FA model have upper critical dimension $d_c = 2$*, *J. Stat. Mech.* (2006) P03006 [cond-mat/0601529].

[50] O. Al Hammal, J.A. Bonachela and M.A. Muñoz, *Absorbing state phase transitions with a non-accessible vacuum*, *J. Stat. Mech.* (2006) P12007 [cond-mat/0610632].

[51] A. Dumitru, E. Iancu, L. Portugal, G. Soyez and D.N. Triantafyllopoulos, *Pomeron loop and running coupling effects in high energy QCD evolution*, *JHEP* **08** (2007) 062 [arXiv:0706.2540].

[52] J.L. Albacete, N. Armesto, J.G. Milhano, C.A. Salgado and U.A. Wiedemann, *Numerical analysis of the Balitsky-Kovchegov equation with running coupling: dependence of the saturation scale on nuclear size and rapidity*, *Phys. Rev.* **D 71** (2005) 014003 [hep-ph/0408216].

[53] S. Bondarenko and L. Motyka, *Solving effective field theory of interacting QCD Pomerons in the semi-classical approximation*, *Phys. Rev.* **D 75** (2007) 114015 [hep-ph/0605185].

[54] A. Schwimmer, *Inelastic rescattering and high-energy reactions on nuclei*, *Nucl. Phys.* **B 94** (1975) 445.

[55] M. Braun, *Structure function of the nucleus in the perturbative QCD with $N(c) \to \infty$ (BFKL Pomeron fan diagrams)*, *Eur. Phys. J.* **C 16** (2000) 337 [hep-ph/0001268].

[56] R. Peschanski, *On the maximal noise for stochastic and QCD traveling waves*, arXiv:0803.2626.

JHEP05(2008)103